Complex Analysis: Final Exam

MartiniPlaza, Wednesday 30 January 2019, 14:00–17:00
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** on the envelope and at the top of each answer sheet.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope). The exam and its solutions will be uploaded to Nestor in the following days.
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.

Question 1 (10 points)

Show that if $f(z)$ and $\overline{f(z)}$ are both analytic in a domain D then $f(z)$ is constant in D.

Question 2 (20 points)

(a) (8 points) Consider the integral

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{-2\mathrm{i}x}}{x+1}\,\mathrm{d}x.$$

Specify and draw the (closed) contour that you should use to compute such an integral with the calculus of residues. Give full justification for your choice of contour.

(b) (12 points) Evaluate the integral

$$\text{pv} \int_{-\infty}^{\infty} \frac{x+1}{(x^2+1)^2}\,\mathrm{d}x,$$

using the calculus of residues. Give complete arguments.

Question 3 (20 points)

Consider the function

$$f(z) = z e^{\mathrm{i}/z^2}.$$

The Laurent series of $f(z)$ for $|z| > 0$ is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where it is given that $c_0 = 0$, $c_1 = 1$, and $c_{-1} = \mathrm{i}$.

(a) (8 points) Compute the rest of the coefficients of the Laurent series of $f(z)$ for $|z| > 0$.

(b) (4 points) Give the type of the singularity of f at $z_0 = 0$ (removable, pole of order m, essential). Justify your answer.

(c) (4 points) Determine the residue of f at $z_0 = 0$. Justify your answer.

(d) (4 points) Determine the domain in which the Taylor series of $f(z)$ at $z_1 = 1+\mathrm{i}$ converges. Justify your answer.

Question 4 (15 points)

(a) (6 points) Let m be a positive integer. Show that for all z on the unit circle we have

$$\left|1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^m}{m!}\right| < e.$$

(b) (9 points) Let m, n be positive integers. Show that the polynomial

$$P(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^m}{m!} + 3z^n$$

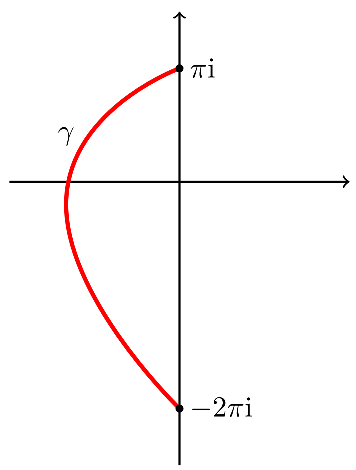
has exactly n zeros inside the unit disk. Give complete arguments.

Question 5 (15 points)

Compute the following integrals along the path γ shown below that lies in the left half-plane, starts at $\pi\mathrm{i}$ and ends at $-2\pi\mathrm{i}$. Give complete arguments.

(a) (6 points) $\displaystyle\int_\gamma z\,\mathrm{d}z$.

(b) (9 points) $\displaystyle\int_\gamma \frac{1}{z}\,\mathrm{d}z$.



Question 6 (10 points)

Answer only one of the following two questions:

Question A. Consider the Möbius transformation

$$f(z) = \frac{2z}{z+1}$$

on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. After computing $f(0)$, $f(\pm 1)$, and $f(\pm \mathrm{i})$, determine the image of the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ under f.

Question B. Given that $f(z)$ is analytic at $z = 0$ and that $f(1/n) = 1/n^4$ for $n = 1, 2, \ldots$, find $f(z)$. Justify your conclusion that your solution $f(z)$ is the *only* function satisfying these properties. Then find a function $g(z)$ which is *not* analytic at $z = 0$ and satisfies $g(1/n) = 1/n^4$ for $n = 1, 2, \ldots$. *Hint:* Find a function $h(z)$ which is not analytic at $z = 0$ and satisfies $h(1/n) = 0$ for $n = 1, 2, \ldots$.

Formulas

The Cauchy-Riemann equations for a function $f = u + \mathrm{i}v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The principal value of the logarithm is

$$\operatorname{Log} z = \operatorname{Log}|z| + \mathrm{i} \operatorname{Arg} z.$$

The generalized Cauchy integral formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

The residue of a function f at a pole z_0 of order m is given by

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z) \right].$$